Received: *September 20, 1999*, Accepted: *February 14, 2000*

Graded Lie algebras and left-right gauge model

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Abstract: We reformulate the left-right gauge model of Pati-Mohapatra using Wulkenhaar's Lie algebraic approach of noncommutative geometry. The mixing angles obtained are $\sin^2\theta_W = 3/8$ and $\sin^2\theta_s = 3/5$, which are identical to the ones predicted in SO(10)-GUT.

Keywords: Spontaneous Symmetry Breaking, Differential and Algebraic Geometry, GUT.

JHEP02(2000)018

Contents

1. Introduction

One of the greatest achievement of noncommutative geometry (abbreviated NCG hereafter) is its geometrization of the standard model [1, 2, 3]. NCG provides a framework where the Higgs boson H may be introduced on the same level as $W^{\pm}$ and Z bosons. In this approach we introduce additional discrete dimensions to the four-dimensional space-time. If the gauge bosons are associated to the continuous directions, the Higgs boson results from gauging the discrete directions. However Connes' prescription is compatible only with linear representations of the matrix group which imposes very stringent constraints on gauge models. Indeed we can show [4, 5] that the only models which can be constructed in this approach are the standard model, the Pati-Salam [7] and the Pati-Mohapatra models [22, 23]. Now, if one takes into account the reality condition of the K-cycle [6], the last two models will be ruled out leaving at the end the standard model as the unique model compatible with Connes' prescription.

The other formulation of NCG due to Coquereaux [8, 9] does not suffer from this problem [10]–[13]. There are some variant of Connes' theory [14, 15] where we use an auxiliary Hilbert space to fit GUT.

R. Wulkenhaar has recently proposed a modification of noncommutative geometry where the differential geometry is formulated in terms of graded differential Lie algebras instead of unital associative $\star$-algebras in the case of Connes' approach. Its application to a list of physical models has been successful. This list includes the standard model [18], the flipped $\mathrm{SU}(5) \times \mathrm{U}(1)$ model [21] and $\mathrm{SO}(10)$ model [20]. The interesting feature of Wulkenhaar's approach is the use of graded Lie algebra as Coquereaux's approach and the obtention of mass relations between fermion and boson masses as Connes' approach.

The gauge models which have not been treated so far are left-right models (Pati-Mohapatra and Pati-Salam models) and the models based on exceptional groups (e.g. $E(6)$).

The purpose of this work is to reconstruct the left-right gauge model (LRM) [22, 23] in the framework of Wulkenhaar's approach of NCG. One of the successes of Connes' and Wulkenhaar's approaches is their geometric explanation of the spontaneous breakdown of gauge symmetry. It is tempting then to investigate whether parity violation in left-right models has also a geometric explanation. Iochum and Schüker [29] have proved that the Connes' approach could not provide an explanation of this violation. In this work the same issue will be discussed using Wulkenhaar's approach.

In section 2 we present the main elements to formulate gauge theories using graded Lie algebras. In section 3 we derive the bosonic and fermionic actions of left-right model and finally we discuss our results.

2. The Lie algebraic formulation of noncommutative geometry

Wulkenhaar's approach is based on the L-cycle $(\mathfrak{g},\mathcal{H}, D, \pi, \Gamma)$ [16, 17]

$$\mathfrak{g} = C^\infty(X) \otimes \mathfrak{a}\,, \tag{2.1}$$

where $C^\infty(X)$ is the algebra of smooth functions on the manifold X and $\mathfrak{a}$ is the matrix Lie algebra.

$\mathcal{H}$ is the Hilbert space

$$\mathcal{H} = \mathbf{L}^2(X, S) \otimes \mathbb{C}^F\,, \tag{2.2}$$

where $\mathbf{L}^2(X, S)$ is the Hilbert space of spinors.

D is the total Dirac operator given by:

$$D = \mathbf{D} \otimes \mathbf{1}_F + \gamma^5 \otimes \mathcal{M}\,, \tag{2.3}$$

where $\mathbf{D}$ and γ^5 are the Dirac operator of the spin connection and the chirality operator on $\mathbf{L}^2(X, S)$ and $\mathcal{M}$ the Dirac operator associated to the discrete algebra $\mathfrak{a}$.

The representation of $\mathfrak{g}$ on $\mathcal{H}$ is given by

$$\pi = \mathbf{1}\otimes\hat{\pi}\,, \tag{2.4}$$

where $\hat{\pi}$ is the representation of $\mathfrak{a}$ on $\mathbb{C}^F$.

Finally, the graded operator Γ, acting on $\mathcal{H}$, is given by:

$$\Gamma = \gamma^5 \otimes \hat{\Gamma}\,, \tag{2.5}$$

where $\hat{\Gamma}$ is the grading operator acting on $\mathbb{C}^F$.

The construction of the universal graded differential Lie algebra proceeds as follows:

- the space $\hat{\Omega}^1\mathfrak{a}$ is generated by the elements of the type

$$\omega^1 = \sum_{\alpha,z}[a_\alpha^z, \ldots, [a_\alpha^1, da_\alpha^0] \ldots], \qquad a_\alpha^i \in \mathfrak{a}\,. \tag{2.6}$$

  where d is the universal differential.

  The representation $\hat{\pi}$ acts on the space $\hat{\Omega}^1\mathfrak{a}$ as:

$$\begin{aligned} \hat{\pi} : \hat{\Omega}^1\mathfrak{a} &\longrightarrow \mathbb{M}_F(\mathbb{C}) \\ \tau^1 = \hat{\pi}(\omega^1) := \sum_{\alpha,z} &\left[\hat{\pi}(a_\alpha^z), \ldots [\hat{\pi}(a_\alpha^1), [-i\mathcal{M}, \hat{\pi}(a_\alpha^0)]] \ldots\right]. \end{aligned} \tag{2.7}$$

  We define also the mapping:

$$\begin{aligned} \hat{\sigma} : \hat{\Omega}^1\mathfrak{a} &\longrightarrow \mathbb{M}_F(\mathbb{C}) \\ \hat{\sigma}(\omega^1) := \sum_{\alpha,z} &\left[\hat{\pi}(a_\alpha^z), \ldots [\hat{\pi}(a_\alpha^1), [\mathcal{M}^2, \hat{\pi}(a_\alpha^0)]] \ldots\right]; \end{aligned} \tag{2.8}$$

- the space $\hat{\Omega}^n\mathfrak{a}$ is spanned by the elements

$$\omega^n = \sum_{\alpha,z}\left[\omega_{n,\alpha}^1, [\omega_{n-1,\alpha}^1, \ldots [\omega_{2,\alpha}^1, \omega_{1,\alpha}^1] \ldots]\right], \tag{2.9}$$

  where $\omega_{i,\alpha}^1 \in \hat{\Omega}^1\mathfrak{a}$.

Similarly we extend $\hat{\pi}$ and $\hat{\sigma}$ recursively to $\hat{\Omega}^n\mathfrak{a}$ by

$$\begin{aligned} \hat{\pi}([\omega^1, \omega^k]) &= \hat{\pi}(\omega^1)\hat{\pi}(\omega^k) - (-1)^k\hat{\pi}(\omega^k)\hat{\pi}(\omega^1) \\ \hat{\sigma}([\omega^1, \omega^k]) &= \hat{\sigma}(\omega^1)\hat{\pi}(\omega^k) - \hat{\pi}(\omega^k)\hat{\sigma}(\omega^1) - \hat{\pi}(\omega^1)\hat{\sigma}(\omega^k) - (-1)^k\hat{\sigma}(\omega^k)\hat{\pi}(\omega^1)\,. \end{aligned}$$

For $k \geqslant 2$ we have

$$\hat{\pi}(J^k\mathfrak{a}) = \left\{\hat{\sigma}(\omega^{k-1}); \omega^{k-1} \in \hat{\Omega}^{k-1}\mathfrak{a} \cap \ker(\hat{\pi})\right\}. \tag{2.10}$$

The main difference between Connes' and Wulkenhaar's approaches to NCG is that in the latter the connection ρ and the curvature θ are not, in general, elements of $\Omega^1\mathfrak{a}$ and $\Omega^2\mathfrak{a}$. To construct connection and curvature we need an additional structure [19].

The spaces $\mathbf{r}^0\mathfrak{a}$, $\mathbf{r}^1\mathfrak{a} \subset M_F(\mathbb{C})$ defined by the conditions [18, 16]:

$$\begin{aligned} \mathfrak{r}^0(\mathfrak{a}) &= \hat{\Gamma}\mathfrak{r}^0(\mathfrak{a})\hat{\Gamma} \\ \left[\mathfrak{r}^0(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\right] &\subset \hat{\pi}(\mathfrak{a}) \\ \left[\mathfrak{r}^0(\mathfrak{a}), \hat{\pi}(\Omega^1\mathfrak{a})\right] &\subset \hat{\pi}(\Omega^1\mathfrak{a}) \\ \left\{\mathfrak{r}^0(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\right\} &\subset \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\} + \hat{\pi}(\Omega^2\mathfrak{a}) \\ \left[\mathfrak{r}^0(\mathfrak{a}), \hat{\pi}(\Omega^1\mathfrak{a})\right] &\subset \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\Omega^1\mathfrak{a})\} + \hat{\pi}(\Omega^3\mathfrak{a}) \end{aligned} \tag{2.11}$$

and

$$
\begin{aligned}
\mathfrak{r}^1(\mathfrak{a}) &= -\hat{\Gamma}\mathfrak{r}^1(\mathfrak{a})\hat{\Gamma} \\
\left[\mathfrak{r}^1(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\right] &\subset \hat{\pi}(\Omega^1\mathfrak{a}) \\
\left\{\mathfrak{r}^1(\mathfrak{a}), \hat{\pi}(\Omega^1\mathfrak{a})\right\} &\subset \hat{\pi}(\Omega^2\mathfrak{a}) + \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\}\,,
\end{aligned}
\tag{2.12}
$$

where $\hat{\Gamma}$ is the grading operator.

The connection is given by:

$$
\rho = \sum_\alpha (c^1_\alpha \otimes m^0_\alpha + c^0_\alpha \gamma^5 \otimes m^1_\alpha)\,, \tag{2.13}
$$

where

$$
c^1_\alpha \in \Lambda^1\,, \qquad c^0_\alpha \in \Lambda^0 \tag{2.14}
$$

Λ^k is the space of k-forms represented by gamma matrices, and

$$
\begin{aligned}
m^0_\alpha &\in \mathfrak{r}^0(\mathfrak{a})\,, \\
m^1_\alpha &\in \mathfrak{r}^1(\mathfrak{a})\,.
\end{aligned}
\tag{2.15}
$$

To construct the curvature θ we need the spaces $\mathbf{j}^0\mathfrak{a}$, $\mathbf{j}^1\mathfrak{a}$ and $\mathbf{j}^2\mathfrak{a}$ which are defined as:

$$
\mathbf{j}^0\mathfrak{a} = \mathbf{c}^0\mathfrak{a}\,, \tag{2.16}
$$

where

$$
\begin{aligned}
\mathbf{c}^0\mathfrak{a} &= \mathbf{c}^0\mathfrak{a}\hat{\Gamma}\mathbf{c}^0\mathfrak{a} \\
\mathbf{c}^0\mathfrak{a}\cdot\hat{\pi}(a) &= 0 \\
\mathbf{c}^0\mathfrak{a}\cdot\hat{\pi}(\Omega^1 a) &= 0
\end{aligned}
\tag{2.17}
$$

and

$$
\mathbf{j}^1\mathfrak{a} = \mathbf{c}^1\mathfrak{a}\,, \tag{2.18}
$$

where

$$
\begin{aligned}
\mathbf{c}^1\mathfrak{a} &= -\mathbf{c}^1\mathfrak{a}\hat{\Gamma}\mathbf{c}^1\mathfrak{a} \\
\mathbf{c}^1\mathfrak{a}\cdot\hat{\pi}(a) &= 0 \\
\mathbf{c}^1\mathfrak{a}\cdot\hat{\pi}(\Omega^1 a) &= 0
\end{aligned}
\tag{2.19}
$$

and

$$
\mathbf{j}^2\mathfrak{a} = \mathbf{c}^2\mathfrak{a} + \hat{\pi}(\mathfrak{J}^2\mathfrak{a}) + \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\}\,, \tag{2.20}
$$

with

$$
\begin{aligned}
\mathbf{c}^2\mathfrak{a} &= \mathbf{c}^2\mathfrak{a}\hat{\Gamma}\mathbf{c}^2\mathfrak{a} \\
\left[\mathbf{c}^2\mathfrak{a}, \hat{\pi}(\mathfrak{a})\right] &= 0 \\
\left[\mathbf{c}^2\mathfrak{a}, \hat{\pi}(\Omega^1\mathfrak{a})\right] &= 0\,.
\end{aligned}
\tag{2.21}
$$

The curvature θ is given by:

$$\theta = \mathbf{d}\rho + \rho^2 - i\{\gamma^5 \otimes \mathcal{M}, \rho\} + \hat{\sigma}_{\mathfrak{g}}(\rho)\gamma^5 + \mathbb{J}^2\mathfrak{g}\,, \tag{2.22}$$

where

$$\mathbb{J}^2\mathfrak{g} = (\Lambda^2 \otimes j^0\mathfrak{a}) \oplus (\Lambda^1\gamma^5 \otimes j^1\mathfrak{a}) \oplus (\Lambda^0 \otimes j^2\mathfrak{a}), \tag{2.23}$$

$\mathbf{d}$ is the exterior differential and $\hat{\sigma}_{\mathfrak{g}}(\rho)$ is the extension of $id \otimes \hat{\sigma}$ to elements of the form (2.13).

Next we choose the representative of the curvature θ, $\varepsilon(\theta)$, orthogonal to $\mathbb{J}^2\mathfrak{g}$, given by:

$$\varepsilon(\theta) = \mathbf{d}\rho + \rho^2 - i\{\gamma^5 \otimes \mathcal{M}, \rho\} + \hat{\sigma}_{\mathfrak{g}}(\rho)\gamma^5 + j\,, \tag{2.24}$$

where

$$\int_X dx\, \mathrm{tr}(\varepsilon(\theta)j') = 0 \quad \forall j' \in \mathbb{J}^2\mathfrak{g}\,. \tag{2.25}$$

with the trace performed both over $\mathbb{M}_F(\mathbb{C})$ and gamma matrices.

The bosonic and the fermionic actions are given by:

$$S_B = \frac{1}{N_F g_0^2}\int_X dx\, \mathrm{tr}(\varepsilon(\theta)^2) \tag{2.26}$$

and

$$S_F = \int_X dx \Psi^*(D + i\rho)\Psi\,, \tag{2.27}$$

where g_0 is a coupling constant.

3. The construction

The discrete L-cycle $(\mathfrak{a}, \mathcal{H}, \mathcal{M})$ consists of the Lie algebra given by:

$$\mathfrak{a} = \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \ni \{a_3, a_2, a_2', a_1\}\,. \tag{3.1}$$

The Hilbert space $\mathcal{H}$ is $\mathbb{C}^{48}$ labelled by the elements

$$(\mathbf{u}_L, \mathbf{d}_L, \mathbf{u}_R, \mathbf{d}_R, \nu_L, e_L, \nu_R, e_R)^T\,,$$

where $\mathbf{u}_L, \mathbf{d}_L, \mathbf{u}_R, \mathbf{d}_R \in \mathbb{C}^3 \otimes \mathbb{C}^3$ and $\nu_L, e_L, \nu_R, e_R \in \mathbb{C}^3$.

The Lie algebra $\mathfrak{a}$ acts on $\mathcal{H}$ via the representation:

$$\hat{\pi}((a_1, a_2, a_2', a_3)) = \begin{bmatrix} \hat{\pi}_q((a_1, a_2, a_2', a_3)) & 0 \\ 0 & \hat{\pi}_l((a_1, a_2, a_2', a_3)) \end{bmatrix}, \tag{3.2}$$

where q and l label, respectively, the quarkionic and the leptonic sectors, with the representation of the quark sector given by:

$$\hat{\pi}_q((a_1, a_2, a_2', a_3)) = \begin{bmatrix} \hat{\pi}_q^1((a_1, a_2, a_2', a_3)) & 0 \\ 0 & \hat{\pi}_q^2((a_1, a_2, a_2', a_3)) \end{bmatrix}, \tag{3.3}$$

where

$$\hat{\pi}_q^1((a_1, a_2, a_2', a_3)) = if_0 \,\mathrm{diag}(\alpha \mathbf{1}_3 \otimes \mathbf{1}_3, \beta \mathbf{1}_3 \otimes \mathbf{1}_3) + \\ + \begin{bmatrix} (a_3 + if_3^1 \mathbf{1}_3) \otimes \mathbf{1}_3 & i(f_1^1 - if_2^1)\mathbf{1}_3 \otimes \mathbf{1}_3 \\ i(f_1^1 + if_2^1)\mathbf{1}_3 \otimes \mathbf{1}_3 & (a_3 - if_3^1 \mathbf{1}_3) \otimes \mathbf{1}_3 \end{bmatrix} \tag{3.4}$$

and

$$\hat{\pi}_q^2((a_1, a_2, a_2', a_3)) = if_0 \,\mathrm{diag}(\gamma \mathbf{1}_3 \otimes \mathbf{1}_3, \delta \mathbf{1}_3 \otimes \mathbf{1}_3) + \\ + \begin{bmatrix} (a_3 + if_3^2 \mathbf{1}_3) \otimes \mathbf{1}_3 & i(f_1^2 - if_2^2)\mathbf{1}_3 \otimes \mathbf{1}_3 \\ i(f_1^2 + if_2^2)\mathbf{1}_3 \otimes \mathbf{1}_3 & (a_3 - if_3^2 \mathbf{1}_3) \otimes \mathbf{1}_3 \end{bmatrix} \tag{3.5}$$

and the representation of the leptonic sector:

$$\hat{\pi}_l((a_1, a_2, a_2', a_3)) = \begin{bmatrix} \hat{\pi}_l^1((a_1, a_2, a_2', a_3)) & 0 \\ 0 & \hat{\pi}_l^2((a_1, a_2, a_2', a_3)) \end{bmatrix}, \tag{3.6}$$

where

$$\hat{\pi}_l^1((a_1, a_2, a_2', a_3)) = if_0 \,\mathrm{diag}(\upsilon \mathbf{1}_3, \varepsilon \mathbf{1}_3) + \\ + \begin{bmatrix} if_3^1 \otimes \mathbf{1}_3 & i(f_1^1 - if_2^1) \otimes \mathbf{1}_3 \\ i(f_1^1 + if_2^1) \otimes \mathbf{1}_3 & -if_3^1 \otimes \mathbf{1}_3 \end{bmatrix} \tag{3.7}$$

and

$$\hat{\pi}_l^2((a_1, a_2, a_2', a_3)) = if_0 \,\mathrm{diag}(\zeta \mathbf{1}_3, \eta \mathbf{1}_3) + \\ \begin{bmatrix} if_3^2 \otimes \mathbf{1}_3 & i(f_1^2 - if_2^2) \otimes \mathbf{1}_3 \\ i(f_1^2 + if_2^2) \otimes \mathbf{1}_3 & -if_3^2 \otimes \mathbf{1}_3 \end{bmatrix}, \tag{3.8}$$

where we have used the fact that elements of $\mathfrak{su}(2)$ algebra have the representation:

$$a_2 = \begin{bmatrix} if_3^1 & i(f_1^1 - if_2^1) \\ i(f_1^1 + if_2^1) & -if_3^1 \end{bmatrix} \\ a_2' = \begin{bmatrix} if_3^2 & i(f_1^2 - if_2^2) \\ i(f_1^2 + if_2^2) & -if_3^2 \end{bmatrix}. \tag{3.9}$$

Remark that we assume $a_2 \neq a_2'$ to make the left and right sectors coupled differently.

The Dirac operator is given by:

$$\mathcal{M} = \begin{bmatrix} \mathcal{M}_q & 0 \\ 0 & \mathcal{M}_l \end{bmatrix}, \tag{3.10}$$

with

$$\mathcal{M}_q = \begin{bmatrix} 0 & 0 & \mathbf{1}_3 \otimes M_u & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{1}_3 \otimes M_d \\ \mathbf{1}_3 \otimes M_u^* & 0 & 0 & 0 \\ 0 & \mathbf{1}_3 \otimes M_d^* & 0 & 0 \end{bmatrix} \tag{3.11}$$

and

$$\mathcal{M}_l = \begin{bmatrix} 0 & 0 & M_\nu & \mathbf{0} \\ 0 & 0 & 0 & M_e \\ M_\nu^* & 0 & 0 & 0 \\ 0 & M_e^* & 0 & 0 \end{bmatrix}, \tag{3.12}$$

where $M_u, M_d, M_\nu, M_e \in M_3(\mathbb{C})$ are the mass matrices of the fermions.

Remark that we have chosen the coefficients associated with $\mathfrak{u}(1)$ algebra arbitrary, we will see when calculating the fermionic action that (as expected) they corresponds to the hypercharges.

The grading operator is given by:

$$\hat{\Gamma} = \mathrm{diag}(-\mathbf{1}_3 \otimes \mathbf{1}_3, -\mathbf{1}_3 \otimes \mathbf{1}_3, \mathbf{1}_3 \otimes \mathbf{1}_3, \mathbf{1}_3 \otimes \mathbf{1}_3, -\mathbf{1}_3, -\mathbf{1}_3, \mathbf{1}_3, \mathbf{1}_3)\,. \tag{3.13}$$

The space $\hat{\pi}(\Omega^1 a)$ is generated by elements of the type

$$\tau^1 = \sum_{\alpha,z} \left[\hat{\pi}(a_\alpha^z), \ldots \left[\hat{\pi}(a_\alpha^1), \left[-i\mathcal{M}, \hat{\pi}(a_\alpha^0)\right]\right] \ldots\right], \tag{3.14}$$

where $a_\alpha^i = (a_1^i, a_2^i, a_2^{i\prime}, a_3^i)$.

Simple calculation gives the following decomposition:

$$\tau^1 = \begin{bmatrix} \tau_q^1 & 0 \\ 0 & \tau_l^1 \end{bmatrix}, \tag{3.15}$$

where for the quark sector we have

$$\tau_q^1 = \begin{bmatrix} 0 & \tau_{1q}^1 \\ \tau_{2q}^1 & 0 \end{bmatrix}, \tag{3.16}$$

with

$$\tau_{1q}^1 = i \begin{bmatrix} \bar{b}_2 \mathbf{1}_3 \otimes M_d + \bar{c}_2 \mathbf{1}_3 \otimes M_u & b_1 \mathbf{1}_3 \otimes M_d + c_1 \mathbf{1}_3 \otimes M_u \\ -\bar{b}_1 \mathbf{1}_3 \otimes M_u - \bar{c}_1 \mathbf{1}_3 \otimes M_d & b_2 \mathbf{1}_3 \otimes M_u + c_2 \mathbf{1}_3 \otimes M_d \end{bmatrix} \tag{3.17}$$

and

$$\tau_{2q}^1 = i \begin{bmatrix} b_2 \mathbf{1}_3 \otimes M_d^* + c_2 \mathbf{1}_3 \otimes M_u^* & -b_1 \mathbf{1}_3 \otimes M_u^* - c_1 \mathbf{1}_3 \otimes M_d^* \\ \bar{b}_1 \mathbf{1}_3 \otimes M_d^* + \bar{c}_1 \mathbf{1}_3 \otimes M_u^* & \bar{b}_2 \mathbf{1}_3 \otimes M_u^* + \bar{c}_2 \mathbf{1}_3 \otimes M_d^* \end{bmatrix}. \tag{3.18}$$

For the leptonic sector we have

$$\tau_l^1 = \begin{bmatrix} 0 & \tau_{1l}^1 \\ \tau_{2l}^1 & 0 \end{bmatrix}, \tag{3.19}$$

where

$$\tau_{1l}^1 = i \begin{bmatrix} \bar{b}_2 \otimes M_e + \bar{c}_2 \otimes M_\nu & b1 \otimes M_e + c_1 \otimes M_\nu \\ -\bar{b}_1 \otimes M_\nu - \bar{c}_1 \otimes M_e & b_2 \otimes M_\nu + c_2 \otimes M_e \end{bmatrix} \tag{3.20}$$

and

$$\tau_{2l}^1 = i \begin{bmatrix} b_2 \otimes M_\nu^* + c_2 \otimes M_e^* & -b_1 \otimes M_\nu^* - c_1 \otimes M_e^* \\ \bar{b}_1 \otimes M_e^* + \bar{c}_1 \otimes M_\nu^* & \bar{b}_2 \otimes M_e^* + \bar{c}_2 \otimes M_\nu^* \end{bmatrix}. \tag{3.21}$$

We can show that the form τ^1 is antihermitean provided that:

$$\alpha + \beta = \gamma + \delta \tag{3.22}$$

and

$$\upsilon + \varepsilon = \zeta + \eta\,. \tag{3.23}$$

Similarly, straightforward calculations give:

$$\hat{\pi}(\Omega^2\mathfrak{a}) \ni \tau^2 = \{\tau^1, \tau^1\} = \begin{bmatrix} \tau_q^2 & 0 \\ 0 & \tau_l^2 \end{bmatrix}, \tag{3.24}$$

with

$$\tau_l^2 = \begin{bmatrix} \tau_{11}^{l2} & \tau_{12}^{l2} & 0 & 0 \\ \tau_{21}^{l2} & \tau_{22}^{l2} & 0 & 0 \\ 0 & 0 & \tau_{33}^{l2} & \tau_{34}^{l2} \\ 0 & 0 & \tau_{43}^{l2} & \tau_{44}^{l2} \end{bmatrix}, \tag{3.25}$$

where the diagonal elements are:

$$\begin{aligned} -\tau_{11}^{l2} &= 2(|b_1|^2 + |c_2|^2) \otimes M_e M_e^* + 2(|c_1|^2 + |b_2|^2) \otimes M_\nu M_\nu^* + \\ &\quad +2(b_1\bar{c}_1 + \bar{c}_2 b_2) \otimes M_e M_v^* + 2(\bar{b}_1 c_1 + c_2\bar{b}_2) \otimes M_\nu M_e^* \\ -\tau_{22}^{l2} &= 2(|c_1|^2 + |b_2|^2) \otimes M_e M_e^* + 2(|b_1|^2 + |c_2|^2) \otimes M_\nu M_\nu^* + \\ &\quad +2(b_1\bar{c}_1 + \bar{c}_2 b_2) \otimes M_e M_v^* + 2(\bar{b}_1 c_1 + c_2\bar{b}_2) \otimes M_\nu M_e^* \\ -\tau_{33}^{l2} &= 2(|c_2|^2 + |c_1|^2) \otimes M_e M_e^* + 2(|b_1|^2 + |b_2|^2) \otimes M_\nu M_\nu^* + \\ &\quad +2(b_1\bar{c}_1 + \bar{c}_2 b_2) \otimes M_e M_v^* + 2(\bar{b}_1 c_1 + c_2\bar{b}_2) \otimes M_\nu M_e^* \\ -\tau_{44}^{l2} &= 2(|b_1|^2 + |b_2|^2) \otimes M_e M_e^* + 2(|c_1|^2 + |c_2|^2) \otimes M_\nu M_\nu^* + \\ &\quad +2(b_1\bar{c}_1 + \bar{c}_2 b_2) \otimes M_e M_v^* + 2(\bar{b}_1 c_1 + c_2\bar{b}_2) \otimes M_\nu M_e^* \end{aligned} \tag{3.26}$$

and the off-diagonal ones:

$$\begin{aligned} -\tau_{12}^{l2} &= 2(\bar{c}_2 c_1 - b_1\bar{b}_2) \otimes M_{\nu e} \\ -\tau_{21}^{l2} &= 2(c_2\bar{c}_1 - \bar{b}_1 b_2) \otimes M_{\nu e} \\ -\tau_{34}^{l2} &= 2(c_1 b_2 - b_1 c_2) \otimes M_{\nu e} \\ -\tau_{43}^{l2} &= 2(\bar{c}_1\bar{b}_2 - \bar{b}_1\bar{c}_2) \otimes M_{\nu e}\,, \end{aligned} \tag{3.27}$$

with

$$\begin{aligned} M_{\nu e} &= M_\nu M_\nu^* - M_e M_e^* \\ M_{\{\nu e\}} &= M_\nu M_\nu^* + M_e M_e^* \end{aligned} \tag{3.28}$$

and similarly

$$\tau_q^2 = \begin{bmatrix} \tau_{11}^{q2} & \tau_{12}^{q2} & 0 & 0 \\ \tau_{21}^{q2} & \tau_{22}^{q2} & 0 & 0 \\ 0 & 0 & \tau_{33}^{q2} & \tau_{34}^{q2} \\ 0 & 0 & \tau_{43}^{q2} & \tau_{44}^{q2} \end{bmatrix}, \tag{3.29}$$

with

$$\begin{aligned}
-\tau^{q2}_{11} &= 2(|b_1|^2+|c_2|^2)\mathbf{1}_3\otimes M_dM_d^* + 2(|c_1|^2+|b_2|^2)\mathbf{1}_3\otimes M_uM_u^* + \\
&\quad +2(b_1\bar{c}_1+\bar{c}_2b_2)\mathbf{1}_3\otimes M_dM_u^* + 2(\bar{b}_1c_1+c_2\bar{b}_2)\mathbf{1}_3\otimes M_uM_d^* \\
-\tau^{l2}_{22} &= 2(|c_1|^2+|b_2|^2)\mathbf{1}_3\otimes M_dM_d^* + 2(|b_1|^2+|c_2|^2)\mathbf{1}_3\otimes M_uM_u^* + \\
&\quad +2(b_1\bar{c}_1+\bar{c}_2b_2)\mathbf{1}_3\otimes M_dM_u^* + 2(\bar{b}_1c_1+c_2\bar{b}_2)\mathbf{1}_3\otimes M_uM_d^* \\
-\tau^{l2}_{33} &= 2(|c_2|^2+|c_1|^2)\mathbf{1}_3\otimes M_dM_d^* + 2(|b_1|^2+|b_2|^2)\mathbf{1}_3\otimes M_uM_u^* + \\
&\quad +2(b_1\bar{c}_1+\bar{c}_2b_2)\mathbf{1}_3\otimes M_dM_u^* + 2(\bar{b}_1c_1+c_2\bar{b}_2)\mathbf{1}_3\otimes M_uM_d^* \\
-\tau^{l2}_{44} &= 2(|b_1|^2+|b_2|^2)\mathbf{1}_3\otimes M_dM_d^* + 2(|c_1|^2+|c_2|^2)\mathbf{1}_3\otimes M_uM_u^* + \\
&\quad +2(b_1\bar{c}_1+\bar{c}_2b_2)\mathbf{1}_3\otimes M_dM_u^* + 2(\bar{b}_1c_1+c_2\bar{b}_2)\mathbf{1}_3\otimes M_uM_d^*
\end{aligned} \tag{3.30}$$

and

$$\begin{aligned}
-\tau^{q2}_{12} &= 2(\bar{c}_2c_1-b_1\bar{b}_2)\mathbf{1}_3\otimes M_{ud} \\
-\tau^{q2}_{21} &= 2(c_2\bar{c}_1-\bar{b}_1b_2)\mathbf{1}_3\otimes M_{ud} \\
-\tau^{q2}_{34} &= 2(c_1b_2-b_1c_2)\mathbf{1}_3\otimes M_{ud} \\
-\tau^{q2}_{43} &= 2(\bar{c}_1\bar{b}_2-\bar{b}_1\bar{c}_2)\mathbf{1}_3\otimes M_{\nu ud}\,,
\end{aligned} \tag{3.31}$$

where

$$\begin{aligned}
M_{ud} &= M_uM_u^* - M_dM_d^* \\
M_{\{ud\}} &= M_uM_u^* + M_dM_d^*\,.
\end{aligned} \tag{3.32}$$

We can show that:

$$\hat{\sigma}(\omega^1) = \begin{bmatrix} \hat{\sigma}_q(\omega^1) & 0 \\ 0 & \hat{\sigma}_l(\omega^1) \end{bmatrix}, \tag{3.33}$$

where

$$\hat{\sigma}_q(\omega^1) = \begin{bmatrix} \hat{\sigma}_{1q} & 0 \\ 0 & \hat{\sigma}_{2q} \end{bmatrix}, \tag{3.34}$$

with

$$\hat{\sigma}_{1q} = i\begin{bmatrix} if^1_3 1_3\otimes M_{ud} & i(f^1_1-if^1_2)1_3\otimes M_{ud} \\ i(f^1_1+if^1_2)1_3\otimes M_{ud} & -if^1_3 1_3\otimes M_{ud} \end{bmatrix} \tag{3.35}$$

and

$$\hat{\sigma}_{2q} = i\begin{bmatrix} if^2_3 1_3\otimes M_{ud} & i(f^2_1-if^2_2)1_3\otimes M_{ud} \\ i(f^2_1+if^2_2)1_3\otimes M_{ud} & -if^2_3 1_3\otimes M_{ud} \end{bmatrix}. \tag{3.36}$$

For the leptonic sector we have

$$\hat{\sigma}_l(\omega^1) = \begin{bmatrix} \hat{\sigma}_{1l} & 0 \\ 0 & \hat{\sigma}_{2l} \end{bmatrix}, \tag{3.37}$$

with

$$\hat{\sigma}_{1l} = i\begin{bmatrix} if^1_3\otimes M_{\nu e} & i(f^1_1-if^1_2)\otimes M_{\nu e} \\ i(f^1_1+if^1_2)\otimes M_{\nu e} & -if^1_3\otimes M_{\nu e} \end{bmatrix} \tag{3.38}$$

and

$$\hat{\sigma}_{2l} = i \begin{bmatrix} if_3^2 \otimes M_{\nu e} & i(f_1^2 - if_2^2) \otimes M_{\nu e} \\ i(f_1^2 + if_2^2) \otimes M_{\nu e} & -if_3^2 \otimes M_{\nu e} \end{bmatrix} . \tag{3.39}$$

After trivial calculation we can show that

$$\tau^2 = \mathrm{diag}(K' \otimes 1_3, K' \otimes 1_3, K' \otimes 1_3, K' \otimes 1_3, K, K, K, K) \bmod (\hat{\sigma}(\Omega^1 a)) \,, \tag{3.40}$$

with

$$K = \hat{\alpha} M_{\{\nu e\}} + \hat{\beta} M_e M_v^* + \hat{\gamma} M_\nu M_e^* \tag{3.41}$$

and

$$K' = \hat{\alpha} M_{\{ud\}} + \hat{\beta} M_d M_u^* + \hat{\gamma} M_u M_d^* \,, \tag{3.42}$$

where

$$\begin{aligned} \hat{\alpha} &= |b_1|^2 + |b_2|^2 + |c_1|^2 + |c_2|^2 \\ \hat{\beta} &= b_1 \bar{c}_1 + b_2 \bar{c}_2 \\ \hat{\gamma} &= \bar{\hat{\beta}} = \bar{b}_1 c_1 + \bar{b}_2 c_2 \,. \end{aligned} \tag{3.43}$$

Using the conditions given by eqs. (2.11) and (2.12), we can show that:

$$\begin{aligned} \mathfrak{r}^0(\mathfrak{a}) &= \hat{\pi}(\mathfrak{a}) \\ \mathfrak{r}^1(\mathfrak{a}) &= \hat{\pi}(\Omega^1 \mathfrak{a}) \end{aligned} \tag{3.44}$$

Let us adopt the following parametrization

$$\sum_\alpha c_\alpha^1 \otimes a_{1\alpha} = \sum_\alpha c_\alpha^1 \otimes i f_{0\alpha} = i\tilde{\mathbf{A}} \in \Lambda^1 \otimes \mathfrak{u}(1) = i\frac{g'}{2}\gamma^\mu C_\mu \tag{3.45}$$

$$\sum_\alpha c_\alpha^1 \otimes a_{3\alpha} = \hat{\mathbf{A}} \in \Lambda^1 \otimes \mathfrak{su}(3) \tag{3.46}$$

$$\begin{aligned} \sum_\alpha c_\alpha^1 \otimes a_{2\alpha} &= A \in \Lambda^1 \otimes \mathfrak{su}(2) \\ &= \begin{bmatrix} \sum_\alpha c_\alpha^1 \otimes if_{3\alpha} & \sum_\alpha c_\alpha^1 \otimes i(f_{1\alpha} - if_{2\alpha}) \\ \sum_\alpha c_\alpha^1 \otimes i(f_{1\alpha} + if_{2\alpha}) & \sum_\alpha c_\alpha^1 \otimes (-if_{3\alpha}) \end{bmatrix} \\ &= \begin{bmatrix} iA_3 & i(A_1 - iA_2) \\ i(A_1 + iA_2) & -iA_3 \end{bmatrix} \end{aligned} \tag{3.47}$$

and

$$\begin{aligned} \Phi_i &= -\sum_\alpha c_\alpha^0 \otimes b_{i\alpha} \in \Lambda^0 \otimes \mathbb{C} \\ \Xi_i &= -\sum_\alpha c_\alpha^0 \otimes c_{i\alpha} \in \Lambda^0 \otimes \mathbb{C} \,. \end{aligned} \tag{3.48}$$

The connection ρ is given by

$$\rho = \begin{bmatrix} \rho_q & 0 \\ 0 & \rho_l \end{bmatrix} , \tag{3.49}$$

where

$$\rho_l = \begin{bmatrix} \rho_l^{A_1} & \rho_l^{H_1} \\ \rho_l^{H2} & \rho_l^{A_2} \end{bmatrix}, \tag{3.50}$$

with

$$\begin{aligned}
\rho_l^{A_1} &= \begin{bmatrix} i(\upsilon\tilde{A} + A_3) \otimes 1_3 & i(A_1 - iA_2) \otimes 1_3 \\ i(A_1 + iA_2) \otimes 1_3 & i(\varepsilon\tilde{A} - A_3) \otimes 1_3 \end{bmatrix} \\
\rho_l^{A_2} &= \begin{bmatrix} i(\zeta\tilde{A} + A'_3) \otimes 1_3 & i(A'_1 - iA'_2) \otimes 1_3 \\ i(A'_1 + iA'_2) \otimes 1_3 & i(\eta\tilde{A} - A'_3) \otimes 1_3 \end{bmatrix} \\
\rho_l^{H_1} &= \begin{bmatrix} -i\gamma^5\bar{\Phi}_2 \otimes M_e - i\gamma^5\bar{\Xi}_2 \otimes M_\nu & -i\gamma^5\Phi_1 \otimes M_e - i\gamma^5\Xi_1 \otimes M_\nu \\ i\gamma^5\bar{\Phi}_1 1_3 \otimes M_\nu + i\gamma^5\bar{\Xi}_1 \otimes M_e & -i\gamma^5\Phi_2 \otimes M_\nu - i\gamma^5\Xi_2 \otimes M_e \end{bmatrix} \\
\rho_l^{H2} &= \begin{bmatrix} -i\gamma^5\Phi_2 \otimes M_e^* - i\gamma^5\Xi_2 \otimes M_\nu^* & i\gamma^5\Phi_1 \otimes M_\nu^* + i\gamma^5\Xi_1 \otimes M_e^* \\ -i\gamma^5\bar{\Phi}_1 \otimes M_e^* - i\gamma^5\bar{\Xi}_1 \otimes M_\nu^* & -i\gamma^5\bar{\Phi}_2 \otimes M_\nu^* - i\gamma^5\bar{\Xi}_2 \otimes M_e^* \end{bmatrix}
\end{aligned} \tag{3.51}$$

and

$$\rho_q = \begin{bmatrix} \rho_q^{A_1} & \rho_q^{H_1} \\ \rho_q^{H2} & \rho_q^{A_2} \end{bmatrix}, \tag{3.52}$$

with

$$\begin{aligned}
\rho_q^{A_1} &= \begin{bmatrix} (\hat{A} + i(\alpha\tilde{A} + A_3)1_3) \otimes 1_3 & i(A_1 - iA_2)1_3 \otimes 1_3 \\ i(A_1 + iA_2)1_3 \otimes 1_3 & (\hat{A} + i(\beta\tilde{A} - A_3)1_3) \otimes 1_3 \end{bmatrix} \\
\rho_q^{A_2} &= \begin{bmatrix} (\hat{A} + i(\gamma\tilde{A} + A'_3)1_3) \otimes 1_3 & i(A'_1 - iA'_2)1_3 \otimes 1_3 \\ i(A'_1 + iA'_2)1_3 \otimes 1_3 & (\hat{A} + i(\delta\tilde{A} - A'_3)1_3) \otimes 1_3 \end{bmatrix} \\
\rho_q^{H_1} &= \begin{bmatrix} -i\gamma^5\bar{\Phi}_2 1_3 \otimes M_d - i\gamma^5\bar{\Xi}_2 1_3 \otimes M_u & -i\gamma^5\Phi_1 1_3 \otimes M_d - i\gamma^5\Xi_1 1_3 \otimes M_u \\ i\gamma^5\bar{\Phi}_1 1_3 \otimes M_u + i\gamma^5\bar{\Xi}_1 1_3 \otimes M_d & -i\gamma^5\Phi_2 1_3 \otimes M_u - i\gamma^5\Xi_2 1_3 \otimes M_d \end{bmatrix} \\
\rho_q^{H_2} &= \begin{bmatrix} -i\gamma^5\Phi_2 1_3 \otimes M_d^* - i\gamma^5\Xi_2 1_3 \otimes M_e^* & i\gamma^5\Phi_1 1_3 \otimes M_u^* + i\gamma^5\Xi_1 1_3 \otimes M_d^* \\ -i\gamma^5\bar{\Phi}_1 1_3 \otimes M_d^* - i\gamma^5\bar{\Xi}_1 1_3 \otimes M_e^* & -i\gamma^5\bar{\Phi}_2 1_3 \otimes M_u^* - i\gamma^5\bar{\Xi}_2 1_3 \otimes M_d^* \end{bmatrix}
\end{aligned} \tag{3.53}$$

We use the parametrization

$$\begin{aligned}
\tilde{\mathbf{A}} &= \mathbf{i}\frac{g'}{2}\gamma^\mu C_\mu\,, \\
\mathbf{A} &= i\frac{g_L}{2}\sum_{a=1}^{3} W_\mu^a\gamma^\mu \otimes \sigma^a\,, \\
\mathbf{A}' &= i\frac{g_R}{2}\sum_{a=1}^{3} W_\mu'^a\gamma^\mu \otimes \sigma^a
\end{aligned} \tag{3.54}$$

and

$$\hat{\mathbf{A}} = i\frac{g_s}{2}\sum_{a=1}^{3} G_\mu^a\gamma^\mu \otimes \lambda^a\,, \tag{3.55}$$

where g', g_L, g_R, g_s are respectively the coupling constants of $\mathfrak{u}(1)_{L-B}$, $\mathfrak{su}(2)_L$, $\mathfrak{su}(2)_R$ and $\mathfrak{su}(3)$ algebras.

Let [24]

$$
\begin{aligned}
B_\mu &= \cos(\theta_W)A_\mu - \sin(\theta_W)Z_\mu \\
W^3_\mu &= \sin(\theta_W)A_\mu + \cos(\theta_W)Z_\mu
\end{aligned}
\tag{3.56}
$$

and

$$
\begin{aligned}
C_\mu &= \cos(\theta_S)B_\mu - \sin(\theta_S)Z'_\mu \\
W'^3_\mu &= \sin(\theta_S)B_\mu + \cos(\theta_S)Z'_\mu \,,
\end{aligned}
\tag{3.57}
$$

where the field B_μ is associated to the group $\mathrm{U}(1)_Y$, and θ_W and θ_S are the mixing angles. Once performing a Wick rotation and by identification with the fermionic action provided by the classical model we obtain [26]:

$$
\varepsilon = \eta = \upsilon = \zeta = -1\,, \tag{3.58}
$$

which are the leptons hypercharges in the left-right model [25].

The same treatment, applied to the quarkionic sector, gives

$$
\alpha = \beta = \gamma = \delta = \frac{1}{3}\,, \tag{3.59}
$$

which are the quark hypercharges [25].

Using the conditions given by eqs. (2.17), (2.19) and (2.21) we can show that

$$
\mathbf{j}^0\mathfrak{a} = 0\,, \qquad \mathbf{j}^1\mathfrak{a} = 0 \tag{3.60}
$$

and

$$
\mathbf{j}^2\mathfrak{a} = \mathrm{diag}(\mathbb{R}\mathbf{I}_{36}, \mathbb{R}\mathbf{I}_{12}) + \hat{\pi}(\mathfrak{J}^2\mathfrak{a}) + \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\}\,. \tag{3.61}
$$

Let the anticommutator

$$
\hat{\psi}(\mathfrak{a}) = \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\} = \mathrm{diag}(\{\hat{\pi}_q(\mathfrak{a}), \hat{\pi}_q(\mathfrak{a})\}\,, \{\hat{\pi}_l(\mathfrak{a}), \hat{\pi}_l(\mathfrak{a})\})\,, \tag{3.62}
$$

where

$$
\{\hat{\pi}_q(a), \hat{\pi}_q(a)\} \ni A_q + \Delta_q\,, \tag{3.63}
$$

with

$$
A_q = \begin{bmatrix} A^1_q & 0 \\ 0 & A^2_q \end{bmatrix} \otimes I_3 \tag{3.64}
$$

where

$$
\begin{aligned}
A^1_q &= i \begin{bmatrix} \lambda^0_1 a_3 + i\theta^0_1 I_3 & (\varpi^1 - i\varpi^2)a_3 + i(\hat{\lambda}^1 - i\hat{\lambda}^2)I_3 \\ (\varpi^1 + i\varpi^2)a_3 + i(\hat{\lambda}^1 + i\hat{\lambda}^2)I_3 & \lambda^0_2 a_3 - i\theta^0_1 I_3 \end{bmatrix} \\
A^2_q &= i \begin{bmatrix} \lambda^0_3 a_3 + i\theta^0_2 I_3 & (\varpi^3 - i\varpi^4)a_3 + i(\hat{\lambda}^3 - i\hat{\lambda}^4)I_3 \\ (\varpi^3 + i\varpi^4)a_3 + i(\hat{\lambda}^3 + i\hat{\lambda}^4)I_3 & \lambda^0_4 a_3 - i\theta^0_2 I_3 \end{bmatrix}
\end{aligned}
\tag{3.65}
$$

and

$$\Delta_q = \mathrm{diag}(\lambda_1 \mathbf{I}_3, \lambda_1 \mathbf{I}_3, \lambda_2 \mathbf{I}_3, \lambda_2 \mathbf{I}_3) \otimes \mathbf{I}_3 \,, \tag{3.66}$$

with the condition

$$\lambda_1^0 + \lambda_2^0 = \lambda_3^0 + \lambda_4^0 \tag{3.67}$$

and

$$\{\hat{\pi}_l(a), \hat{\pi}_l(a)\} \ni A_l + \Delta_l \,, \tag{3.68}$$

with

$$A_l = i \begin{bmatrix} i\kappa_1^0 I_3 & i(\check{\lambda}^1 - i\check{\lambda}^2) I_3 & 0 & 0 \\ i(\check{\lambda}^1 + i\check{\lambda}^2) I_3 & -i\kappa_1^0 I_3 & 0 & 0 \\ 0 & 0 & i\kappa_2^0 I_3 & i(\check{\lambda}^3 - i\check{\lambda}^4) I_3 \\ 0 & 0 & i(\check{\lambda}^3 + i\check{\lambda}^4) I_3 & -i\kappa_2^0 I_3 \end{bmatrix} \tag{3.69}$$

and

$$\Delta_l = diag(\chi \mathbf{I}_3, \chi \mathbf{I}_3, \chi \mathbf{I}_3, \chi \mathbf{I}_3) \,, \tag{3.70}$$

with $\lambda_i, \lambda_i^0, \check{\lambda}^i, \hat{\lambda}^i, \theta_j^0, \kappa_j^0, \varpi^i, \chi$ $(i = 1, \ldots, 4,\ j = 1, \ldots, 2) \in \mathbb{R}$.

Trivial calculations give [26]

$$\mathbf{j}^2 a = \hat{\pi}(\mathcal{J}^2 \mathfrak{a}) \oplus \mathrm{diag}(A_q, A_l) \oplus \mathrm{diag}(K_q, K_l) \tag{3.71}$$

where

$$\begin{aligned} K_q &= \mathrm{diag}(\varphi_1 \mathbf{I}_3, \varphi_1 \mathbf{I}_3, \varphi_2 \mathbf{I}_3, \varphi_2 \mathbf{I}_3) \otimes \mathbf{I}_3 \\ K_l &= \mathrm{diag}(\varphi_3 \mathbf{I}_3, \varphi_3 \mathbf{I}_3, \varphi_3 \mathbf{I}_3, \varphi_3 \mathbf{I}_3) \,, \end{aligned} \tag{3.72}$$

with φ_1, φ_2 et $\varphi_3 \in \mathbb{R}$.

The representative $\varepsilon(\{\tau^1, \tau^1\})$ of $\{\tau^1, \tau^1\} + \mathbf{j}^2 a$, orthogonal to $\mathbf{j}^2 a$, is given by:

$$\varepsilon\left(\{\tau^1, \tau^1\}\right) = \mathrm{diag}\left(\tilde{K}' \otimes 1_3, \tilde{K}' \otimes 1_3, \tilde{K}' \otimes 1_3, \tilde{K}' \otimes 1_3, \tilde{K}, \tilde{K}, \tilde{K}, \tilde{K}\right) \bmod (\hat{\sigma}(\Omega^1 a)) \,, \tag{3.73}$$

with

$$\begin{aligned} \tilde{K} &= \hat{\alpha} \tilde{M}_{\{\nu e\}} + \hat{\beta} \widetilde{M_e M_v^*} + \hat{\gamma} \widetilde{M_\nu M_e^*} \\ \tilde{K}' &= \hat{\alpha} \tilde{M}_{\{ud\}} + \hat{\beta} \widetilde{M_d M_u^*} + \hat{\gamma} \widetilde{M_u M_d^*} \end{aligned} \tag{3.74}$$

where

$$\begin{aligned} \tilde{M}_{\{\nu e\}} &= M_\nu M_\nu^* + M_e M_e^* - \frac{1}{3} \mathrm{tr}(M_\nu M_\nu^* + M_e M_e^*) \mathbf{1}_3 \,, \\ \widetilde{M_e M_v^*} &= M_e M_v^* - \frac{1}{3} \mathrm{tr}(M_e M_v^*) \mathbf{1}_3 \,, \\ \widetilde{M_\nu M_e^*} &= M_\nu M_e^* - \frac{1}{3} \mathrm{tr}(M_\nu M_e^*) \mathbf{1}_3 \,, \\ \tilde{M}_{\{ud\}} &= M_u M_u^* + M_d M_d^* - \frac{1}{3} \mathrm{tr}(M_u M_u^* + M_d M_d^*) \mathbf{1}_3 \,, \\ \widetilde{M_d M_u^*} &= M_d M_u^* - \frac{1}{3} \mathrm{tr}(M_d M_u^*) \mathbf{1}_3 \,, \end{aligned} \tag{3.75}$$

and

$$\widetilde{M_u M_d^*} = M_u M_d^* - \frac{1}{3} tr(M_u M_d^*) \mathbf{1}_3 \,.$$

Remark also that

$$\mathcal{M}^2 = \frac{1}{2} \operatorname{diag}\Big(1_3 \otimes \tilde{M}_{\{ud\}}, 1_3 \otimes \tilde{M}_{\{ud\}}, 1_3 \otimes \tilde{M}_{\{ud\}}, 1_3 \otimes \tilde{M}_{\{ud\}},$$
$$\tilde{M}_{\{\nu e\}}, \tilde{M}_{\{\nu e\}}, \tilde{M}_{\{\nu e\}}, \tilde{M}_{\{\nu e\}}\Big) \bmod (\hat{\sigma}(\Omega^1 a)) \,. \tag{3.76}$$

The bosonic action is given by

$$S_B = \int_X (\mathfrak{L}_0 + \mathfrak{L}_1 + \mathfrak{L}_2) \, dx \tag{3.77}$$

where

$$\mathfrak{L}_0 = \frac{1}{96 g_0^2} \Big\{ \operatorname{tr}\left(12 \tilde{M}^2_{\{ud\}} + 4 \tilde{M}^2_{\{\nu e\}}\right) \times \left(|\Phi_1|^2 + |\Phi_2|^2 + |\Xi_1|^2 + |\Xi_2 + 1|^2 - 1\right)^2 +$$
$$+ \operatorname{tr}\left(12 (\widetilde{M_d M_u^*})^2 + 4 (\widetilde{M_e M_\nu^*})^2\right) \times \left(\Phi_1 \bar{\Xi}_1 + \Phi_2 (\bar{\Xi}_2 + 1)\right) +$$
$$+ \operatorname{tr}\left(12 (\widetilde{M_u M_d^*})^2 + 4 (\widetilde{M_\nu M_e^*})^2\right) \times \left(\bar{\Phi}_1 \Xi_1 + \bar{\Phi}_2 (\Xi_2 + 1)\right) \Big\} \tag{3.78}$$

is the quartic Higgs potential and

$$\mathfrak{L}_2 = \frac{1}{4 g_0^2} \operatorname{tr}_c \left(\mathbf{d}\hat{\mathbf{A}} + \frac{1}{2} \left\{ \hat{\mathbf{A}}, \hat{\mathbf{A}} \right\} \right)^2 + \frac{1}{4 g_0^2} \operatorname{tr}_c \left(\mathbf{d}\mathbf{A} + \frac{1}{2} \left\{ \mathbf{A}, \mathbf{A} \right\} \right)^2 +$$
$$+ \frac{1}{4 g_0^2} \operatorname{tr}_c \left(\mathbf{d}\mathbf{A}' + \frac{1}{2} \left\{ \mathbf{A}', \mathbf{A}' \right\} \right)^2 + \frac{1}{3 g_0^2} \operatorname{tr}_c \left(\mathbf{d}\tilde{\mathbf{A}} \right)^2 \tag{3.79}$$

is the Yang-Mills lagrangean.

Since $\mathfrak{L}_1$, the interaction Higgs-boson, is quite long it is not displayed here in its totality [26]. The parts of $\mathfrak{L}_1$ which give rise to the masses of W_R and W_L bosons are respectively:

$$\mathcal{L}_1^1 = \frac{1}{8 g_0^2} \mathbf{m}^2 \operatorname{tr}_c \left| i (A_1 - i A_2)(\Xi_2 + 1) \right|^2 \tag{3.80}$$

and

$$\mathcal{L}_1^2 = \frac{1}{8 g_0^2} \mathbf{m}^2 \operatorname{tr}_c \left| i (A_1' - i A_2')(\Xi_2 + 1) \right|^2 , \tag{3.81}$$

where

$$\mathbf{m}^2 = \operatorname{tr}\left(\frac{1}{3} M_e M_e^* + \frac{1}{3} M_\nu M_\nu^* + M_u M_u^* + M_d M_d^* \right) . \tag{3.82}$$

Let us make the following rescaling

$$\Xi_2 = \frac{\bar{g} \phi_2}{\mathbf{m}} , \tag{3.83}$$

where $\bar{g}$ is a coupling constant which is specified later.

Since

$$i(A_1 - iA_2) = i\frac{g_L}{\sqrt{2}} W^{\dagger}_{\mu L}\gamma^{\mu}$$
$$i(A'_1 - iA'_2) = i\frac{g_L}{\sqrt{2}} W^{\dagger}_{\mu R}\gamma^{\mu} \tag{3.84}$$

and

$$\mathrm{tr}_c(\gamma^{\mu}\gamma^{\nu}) = 4\delta^{\mu\nu} \tag{3.85}$$

then

$$\mathcal{L}_1^1 = \frac{1}{4}\left(\frac{\bar{g}g_L}{g_0}\right)^2 W^{\dagger}_{\mu L}W^{\mu}_L\phi_2^2 + \frac{1}{2}\left(\frac{g_L}{g_0}\right)^2 \mathbf{m}\bar{g}W^{\dagger}_{\mu L}W^{\mu}_L\phi_2 + \frac{1}{4}\left(\frac{g_L}{g_0}\right)^2 \mathbf{m}^2 W^{\dagger}_{\mu L}W^{\mu}_L\,. \tag{3.86}$$

The last term gives:

$$M_{W_L} = \frac{1}{2}\left(\frac{g_L}{g_0}\right)\mathbf{m}\,. \tag{3.87}$$

We can show also that:

$$M_{W_R} = \frac{1}{2}\left(\frac{g_R}{g_0}\right)\mathbf{m}\,. \tag{3.88}$$

Hence

$$\frac{M_{W_L}}{M_{W_R}} = \frac{g_L}{g_R}\,. \tag{3.89}$$

In the case of non-symmetric left-right model

$$g_L \neq g_R\,, \tag{3.90}$$

we have

$$M_{W_L} \neq M_{W_R} \tag{3.91}$$

and the left-handed and the right-handed gauge bosons acquire different masses. It is not the case for the symmetric model, leading to the conclusion that parity violation does not occur for such a model [27, 28]. The same conclusion has been obtained by [29] when discussing the $\mathrm{U}(2)_L\times\mathrm{U}(2)_R$ model in the framework of noncommutative geometry à la Connes-Lott.

For left-right symmetric model Wulkenhaar's approach to noncommutative geometry does not provide a geometric explanation for a parity violation to be contrasted to Coquereaux approach [25].

Remark that if we neglect the mass of fermions in respect of the mass of the top quark m_t we obtain

$$M_{W_L} = \frac{1}{2}\left(\frac{g_L}{g_0}\right)m_t \tag{3.92}$$

and

$$M_{W_R} = \frac{1}{2}\left(\frac{g_R}{g_0}\right)m_t\,. \tag{3.93}$$

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The masses Z et Z' bosons come from

$$\mathcal{L}_1^3 = \frac{1}{8g_0^2}\mathbf{m}^2 \operatorname{tr}_c |i(A_0 - iA_3)(\Xi_2 + 1)|^2 \tag{3.94}$$

and

$$\mathcal{L}_1^4 = \frac{1}{8g_0^2}\mathbf{m}^2 \operatorname{tr}_c |i(A_0 - iA_3')(\Xi_2 + 1)|^2 . \tag{3.95}$$

The calculations give

$$M_Z = \frac{\mathbf{m}}{2}\left(\frac{g_L}{g_0}\right)\frac{1}{\cos(\theta_W)} . \tag{3.96}$$

Since

$$M_{W_L} = \frac{1}{2}\left(\frac{g_L}{g_0}\right)\mathbf{m} , \tag{3.97}$$

then

$$M_Z = \frac{M_{W_L}}{\cos(\theta_W)} , \tag{3.98}$$

which is the same relation as in the standard model.

Y. Okumura [25] has found the same relation when applying noncommutative geometry à la Coquereaux to left-right gauge model.

Similarly, we obtain

$$M_{z'} = \frac{\mathbf{m}}{2}\left(\frac{g'}{g_0}\right)\sin(\theta_s) . \tag{3.99}$$

We can show also that the photon and gluons remain massless.

From eq (3.78) the free lagrangean associated with the field Ξ_2 is given by:

$$\begin{aligned}\mathfrak{L}_{free}^{\Xi_2} &= \frac{1}{8g_0^2}\Upsilon^2(|\Xi_2 + 1|^2 - 1)^2 \\ &= \frac{1}{2}\left(\frac{\bar{g}}{g_0}\right)^2\left(\frac{\Upsilon}{\mathbf{m}}\right)^2\phi_2^2 + \text{higher order terms} ,\end{aligned} \tag{3.100}$$

where

$$\Upsilon^2 = tr(\tilde{M}_{\{ud\}}^2 + \frac{1}{3}\tilde{M}_{\{\nu e\}}^2) . \tag{3.101}$$

Hence

$$m_{\phi_2} = \left(\frac{\bar{g}}{g_0}\right)\left(\frac{\Upsilon}{\mathbf{m}}\right) . \tag{3.102}$$

If we take $\bar{g} = g_0$, then

$$m_{\phi_2} = \sqrt{\frac{\operatorname{tr}\left(\tilde{M}_{\{ud\}}^2 + \frac{1}{3}\tilde{M}_{\{\nu e\}}^2\right)}{\operatorname{tr}\left(M_u M_u^* + M_d M_d^* + \frac{1}{3}M_e M_e^* + \frac{1}{3}M_\nu M_\nu^*\right)}} . \tag{3.103}$$

Neglecting the mass of fermions in respect to the mass of the top quark gives:

$$m_{\phi_2} \approx \sqrt{\frac{\mathrm{tr}(\tilde{M}^2_{\{ud\}})}{\mathrm{tr}(M_u M_u^*)}} = \frac{3}{2} m_t \,, \tag{3.104}$$

which is the same relation found by Wulkenhaar for the neutrinos mass matrix $M_\nu \neq 0$ [18].

Let us now turn to the mixing angles, eq (3.79) looks like its counterpart in classical model provided that:

$$g_s = g_L = g_R = g_0 \tag{3.105}$$

and

$$g' = \sqrt{\frac{3}{2}} g_0 \,, \tag{3.106}$$

which give:

$$\sin^2(\theta_W) = \frac{3}{8} \tag{3.107}$$

and

$$\sin^2(\theta_S) = \frac{3}{5} \,, \tag{3.108}$$

which coincide with SO(10)-GUT prediction. However du to eq. (3.106) the grand unification for coupling constants is not achieved.

4. Conclusion

We reformulate the left-right model using Wulkenhaar's approach to noncommutative geometry. We have found that the left-right symmetric model does not exhibit the parity violation, however this approach gives the mixing angles predicted by SO(10)-GUT and at the tree level we have the mass relations

$$M_{W_L} = \frac{1}{2} \left(\frac{g_L}{g_0} \right) m_t \tag{4.1}$$

$$M_{W_R} = \frac{1}{2} \left(\frac{g_R}{g_0} \right) m_t \tag{4.2}$$

$$M_Z = \frac{M_{W_L}}{\cos(\theta_W)} \tag{4.3}$$

$$M_{Z'} = \frac{g' \sin(\theta_S)}{g_R} M_{W_R} \tag{4.4}$$

and

$$M_H = \frac{3}{2} m_t \,, \tag{4.5}$$

where m_t is the top mass.

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In Wulkenhaar's approach to NCG the unification of the coupling constants is not achieved.

The main result of this work is that Wulkenhaar's (like Connes') approach to noncommutative geometry could not explain the parity breaking in the left-right model.

Acknowledgments

One of us (A.B.) would like to express his sincere thanks to Dr. R. Wulkenhaar and Dr. F. Cuypers for private communications.

This work was supported by the Algerian Ministère de l'Enseignement Supérieur et La Recherche Scientifique under the contract D2501/01/05/97.

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